110 points

Math 112
Test 4

PLEDGED Key

Use your own paper. Put problems in order.

1. Given: $f(x)=\sqrt[3]{x-4}$. Find the following: (20 points)
   a) Find the first 5 nonzero terms in the Taylor series about c = -4.
   b) Find an expression for the series.

2. State and prove Taylor's Theorem. Clearly define each step in the induction process. (20 points)

3. Use a geometric series to find a series representation for $f(x)=\dfrac{x^2}{4+25x^2}$ about c =0. Give the domain for this series. (10 points)

4. Change from series notation to functional notation for $\sum_{n=0}^{\infty}\dfrac{(-1)^n(x+2)^{n+1}}{n+1}$. Give the domain. (10 points)

5. Use Taylor's Theorem to derive a series representation for $f(x)=\sin(x)$ Use this result to find a series representation for: $g(x)=\dfrac{x}{2}\sin\left(\dfrac{x^2}{2}\right)$. Give the domain for g. (15 points)

6. (20 points) Find the radius and interval of convergence for: $\sum_{k=2}^{\infty}\dfrac{(-1)^k(x-3)^k}{k\ln(k)}$

7. (15 points) Find the radius of convergence for: $\sum_{n=1}^{\infty}\dfrac{n!x^n}{(2n)^n}$

1.) $f(x) = (x-4)^{1/3}$

$c = -4$

a) $f(x) = (x-4)^{1/3}$ $\qquad f(-4) = -2$

$f'(x) = \frac{1}{3}(x-4)^{-2/3}$ $\qquad f'(-4) = \frac{1}{3} \cdot \frac{1}{(-2)^2}$

$f''(x) = \frac{1}{3} \cdot \frac{-2}{3}(x-4)^{-5/3}$ $\qquad f''(-4) = \frac{-2}{3^2} \cdot \frac{1}{(-2)^5}$

10 $f'''(x) = \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{5}{3}(x-4)^{-8/3}$ $\qquad f'''(-4) = \frac{2 \cdot 5}{3^3} \cdot \frac{1}{(-2)^8}$

$f^{(4)}(x) = \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{5}{3} \cdot \frac{-8}{3}(x-4)^{-11/3}$ $\qquad f^{(4)}(-4) = \frac{-2 \cdot 5 \cdot 8}{3^4} \cdot \frac{1}{(-2)^{11}}$

$$f(x) = -2 + \frac{1}{12}(x-4) + \frac{2}{3^2(2)^5 2!}(x-4)^2 + \frac{2 \cdot 5}{3^3 2^8 3!}(x-4)^2 + \frac{2 \cdot 5 \cdot 8}{3^4 2^{11}}(x-4)^3$$

b)

10 $$-2 + \frac{(x-4)}{12} + \sum_{n=2}^{\infty} \frac{2 \cdot 5 \cdot 8 \cdots (3n-4)}{3^n 2^{3n-1} n!}(x-4)^n$$

2.) Proof - Taylors

20 class notes...

3.) $f(x) = \dfrac{x^2}{4+25x^2} = \dfrac{x^2}{4}\left(\dfrac{1}{1+\frac{25x^2}{4}}\right)$

$f(x) = \dfrac{x^2}{4}\left(\dfrac{1}{1-\left(-\frac{25x^2}{4}\right)}\right) = \dfrac{x^2}{4}\sum_{n=0}^{\infty}\left(-\dfrac{25x^2}{4}\right)^n$

10 $= \dfrac{x^2}{4}\sum_{n=0}^{\infty}\dfrac{(-1)^n 5^{2n} x^{2n}}{4^n} = \boxed{\sum_{n=0}^{\infty}\dfrac{(-1)^n 5^{2n} x^{2n+2}}{4^{n+1}}}$

$\left|\dfrac{25x^2}{4}\right| < 1 \rightarrow |x^2| < \dfrac{4}{25} \rightarrow |x| < \dfrac{2}{5}$

$\boxed{D: -\frac{2}{5} < x < \frac{2}{5}}$

4. $\sum_{n=0}^{\infty}\dfrac{(-1)^n (x+2)^{n+1}}{n+1}$ $\quad c = -2$

differentiate: $\sum_{n=0}^{\infty}\dfrac{(-1)^n \cancel{(n+1)} (x+2)^n}{\cancel{(n+1)}} = \sum_{n=0}^{\infty}(-1)^n (x+2)^n$

$\dfrac{1}{1-(-(x+2))} = \dfrac{1}{1+x+2} = \dfrac{1}{3+x}$

integrate: $\displaystyle\int \dfrac{dx}{3+x} = \ln|3+x| + C$

center $= -2$ so $f(-2) = \sum_{n=0}^{\infty}\dfrac{(-1)^n (x+2)^{n+1}}{n+1} + C = \ln 1$

$C = 0$

$\ln|3+x|$ $\quad D: |x+2| < 1$

Domain: $-3 < x \leq -1$ (need to show)

5. $f(x) = \sin x \qquad c = 0$

15

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \end{aligned}$$

$$f(x) = 0 + \frac{1}{1!}x + 0 - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

$$f(x) = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sin x}$$

$$g(x) = \frac{x}{2}\sin\left(\frac{x^2}{2}\right) = \frac{x}{2} f\left(\frac{x^2}{2}\right) = \frac{x}{2}\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x^2}{2}\right)^{2n+1}}{(2n+1)!}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{2^{2n+2}(2n+1)!}}$$

$$\text{RT, } \lim_{n\to\infty}\left| \frac{x^4 \cdot \cancel{x^{4n+3}}}{2^2 \cdot \cancel{2^{2n+2}}(2n+3)(2n+2)\cancel{(2n+1)!}} \cdot \frac{\cancel{2^{2n+2}}\cancel{(2n+1)!}}{\cancel{x^{4n+3}}} \right|$$

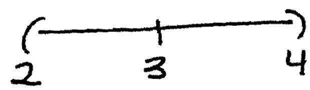
$$\frac{|x^4|}{4} \lim_{n\to\infty} \cancel{\frac{1}{(2n+3)(2n+2)}} \nearrow 0 < 1$$

$\boxed{\text{So, D: all } \mathbb{R}}$

6.) $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k (x-3)^k}{k \ln k}$ By Ratio Test,

2^0 $\displaystyle\lim_{k\to\infty} \left| \frac{(x-3)\cancel{(x-3)^k}}{(k+1)\ln(k+1)} \cdot \frac{k \ln k}{\cancel{(x-3)^k}} \right| < 1$ for convergence

$\displaystyle |x-3| \lim_{k\to\infty} \overset{1}{\cancel{\frac{k}{k+1}}} \cdot \overset{1}{\cancel{\frac{\ln k}{\ln(k+1)}}} < 1 \to |x-3| < 1$



<u>At $x=2$:</u> $(2-3)^k = (-1)^k$

$$\sum_{k=2}^{\infty} \frac{(-1)^k(-1)^k}{k \ln k} = \sum_{k=2}^{\infty} \frac{(-1)^{2k}}{k \ln k} = \sum_{k=2}^{\infty} \frac{1}{k \ln k}$$

By integral test, $\left[\int \frac{dx}{x \ln x} \quad \begin{array}{l} u = \ln x \\ dv = \frac{1}{x} dx \end{array} \right.$

$\left. \int \frac{1}{u} du = \ln|u| + C \to \ln|\ln x| + C \right]$

$$\int_2^{\infty} \frac{dx}{x \ln x} \Rightarrow \lim_{k\to\infty} \int_2^k \frac{dx}{x \ln x} \Rightarrow \lim_{k\to\infty} \ln|\ln x| \Big|_2^k$$

$\displaystyle\lim_{k\to\infty} \ln|\ln x| = +\infty$, diverges

<u>At $x=4$:</u> $(4-3)^k = 1^k = 1$

$\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{k \ln k}$ By AST, $0 < \frac{1}{(k+1)\ln(k+1)} < \frac{1}{k \ln k}$

and $\displaystyle\lim_{k\to\infty} \frac{1}{k \ln k} = 0$ so converges

$$\boxed{2 < x \leq 4}$$

7.) $\displaystyle\sum_{n=1}^{\infty} \frac{n!\,x^n}{(2n)^n}$ By Ratio Test,

$$\lim_{n\to\infty}\left|\frac{(n+1)\cancel{n!}\,x\,\cancel{x^n}}{(2n+2)^{n+1}}\cdot\frac{(2n)^n}{\cancel{n!}\,\cancel{x^n}}\right| < 1$$

15 $$|x|\lim_{n\to\infty}\cancelto{\frac{1}{2}}{\frac{n+1}{(2n+2)}}\cdot\left(\frac{2n}{2n+2}\right)^n < 1$$

Aside

$$\frac{\cancel{2^n}\,n^n}{\cancel{2^n}(n+1)^n}$$

$$\frac{|x|}{2}\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n < 1$$

Aside

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = e^{-1}$$

"1^∞"

$$y=\left(\frac{n}{n+1}\right)^n$$

$$\ln y = n\ln\left(\frac{n}{n+1}\right)$$

$$\lim_{n\to\infty}\left[\frac{\ln\left(\frac{n}{n+1}\right)}{\frac{1}{n}}\right] \overset{L'H}{=} \lim_{n\to\infty}\frac{\left(\frac{\cancel{n+1}}{n}\right)\left(\frac{\cancel{n}+1-\cancel{n}}{(n+1)^{\cancel{2}}}\right)}{-\frac{1}{n^2}} \overset{Alg}{=}$$

"$\frac{0}{0}$"

$$\lim_{n\to\infty}\frac{-n^2}{n(n+1)} = -1$$

$$\frac{|x|}{2}\cdot\frac{1}{e} < 1$$

$|x| < 2e$ so $\boxed{\text{radius} = 2e}$