Math 112 Test 3 PLEDGED Key

Spring, 2010

Use your own paper, put all problems in order, be neat and clear with your work.

1. Proofs: (12 points each)

   a) A geometric series, $\sum_{n=0}^{\infty} a_n r^n$ with ratio of r, converges if $|r| < 1$ with sum of $\frac{a}{1-r}$.

   b) State and prove the nth term test.

2. Sequence: (12 points) Given: $a_n = \cos\left(\frac{\pi}{n}\right)$

   a) Give the first four terms.

   b) Sketch the graph.

   c) Does the sequence, $\{a_n\}$ converge? Explain your reasoning.

3. Find the sum or explain why the series is divergent: (7 points each)

   a) $\sum_{n=0}^{\infty} \frac{6}{n^2 + 5n + 4}$ b) $\sum_{n=2}^{\infty} \frac{(-1)^n 3^{2n} 4^{n-1}}{2^{3n-1} 6^{n+1}}$

4. Test for convergence. Clearly state your argument. If appropriate, tell whether the convergence is conditional or absolute. (12 points each, 60 points)

   a) $\sum_{n=1}^{\infty} \frac{(-1)^n 2^n n!}{[2 \cdot 5 \cdot 8 \cdots (3n-1)]}$ b) $\sum_{k=2}^{\infty} \frac{(-1)^n}{k\sqrt{\ln k}}$

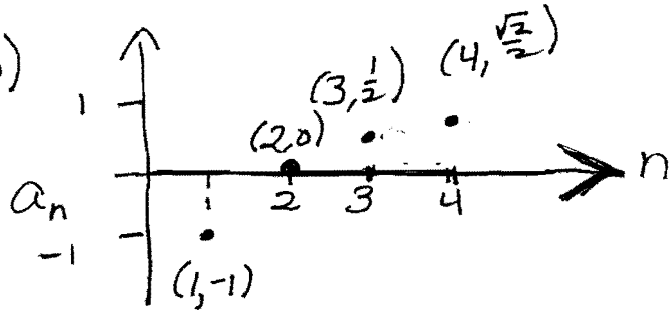
   c) $\sum_{n=1}^{\infty} \frac{Arc\tan(4n)}{n}$ d) $\sum_{n=2}^{\infty} \frac{(-1)^n \sqrt{n-1}}{\sqrt[3]{8n^4 - n^3 + 1}}$

   e) $\sum_{k=1}^{\infty} \frac{2^k}{e^k - k}$

1. a) geometric $\Big\}$ See class notes
(12ea) b) n^{th} term

2.) $a_n = \cos\left(\frac{\pi}{n}\right)$, $n = 1, 2, 3 \cdots$

a) $a_1 = \cos(\pi) = -1$
$a_2 = \cos\left(\frac{\pi}{2}\right) = 0$
$a_3 = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$
(4ea) $a_4 = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

b)


c) $\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right) = 1$, sequence converges

3.) a) $\sum_{n=0}^{\infty} \frac{6}{n^2+5n+4} = \sum_{n=0}^{\infty}\left[\frac{2}{n+1} - \frac{2}{n+4}\right]$
$(n+4)(n+1)$

$S_n = \left(2-\cancel{\frac{2}{4}}\right)+\left(1-\cancel{\frac{2}{5}}\right)+\left(\frac{2}{3}-\cancel{\frac{2}{6}}\right)+\left(\cancel{\frac{2}{4}}-\cancel{\frac{2}{7}}\right)+\left(\cancel{\frac{2}{5}}-\cancel{\frac{2}{8}}\right)+\cdots$

(7ea) $\left(\cancel{\frac{2}{n-2}}-\cancel{\frac{2}{n+1}}\right)+\left(\cancel{\frac{2}{n-1}}-\frac{2}{n+2}\right)+\left(\cancel{\frac{2}{n}}-\frac{2}{n+3}\right)+\left(\cancel{\frac{2}{n+1}}-\frac{2}{n+4}\right)$

$\lim_{n\to\infty}\left(2+1+\frac{2}{3}-\cancel{\frac{2}{n+2}}^{\,0}-\cancel{\frac{2}{n+3}}^{\,0}-\cancel{\frac{2}{n+4}}^{\,0}\right) = \frac{11}{3}$

b) $\sum_{n=2}^{\infty} \frac{(-1)^n 3^{2n} 4^{n-1}}{2^{3n-1} 6^{n+1}} = \sum_{n=2}^{\infty}\left[\frac{(-1)\cancel{9}^{\,3}\cdot\cancel{4}}{\cancel{8}_{\,2}\cdot\cancel{16}_{\,2}}\right]^n\left(\frac{1}{4}\cdot 2\cdot\frac{1}{6}\right)$

$\frac{1}{12}\sum_{n=2}^{\infty}\left(\frac{-3}{4}\right)^n \Rightarrow a = \left(\frac{-3}{4}\right)^2$

$\text{Sum} = \frac{1}{12}\left[\frac{\frac{9}{16}}{1-\left(-\frac{3}{4}\right)}\right] = \frac{1}{\cancel{12}_{\,4}}\cdot\frac{\cancel{9}^{\,3}}{\cancel{16}_{\,4}}\cdot\frac{\cancel{4}}{7} = \frac{3}{4\cdot 28} = \frac{3}{112}$

4. a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 2^n n!}{[2\cdot 5\cdot 8\cdots(3n-1)]}$

(12 ea)

By ratio test,

$$\lim_{n\to\infty}\left|\frac{2\cdot\cancel{2^n}(n+1)\cancel{n!}}{[\cancel{2\cdot5\cdot8\cdots(3n-1)}(3n+2)]}\cdot\frac{\cancel{2\cdot5\cdot8\cdots(3n-1)}}{\cancel{2^n}\,\cancel{n!}}\right| = \frac{2}{3} < 1$$

Converges absolutely

b) $\displaystyle\sum_{k=2}^{\infty}\frac{(-1)^n}{k\sqrt{\ln k}}$ By AST, $0 < \dfrac{1}{(k+1)\sqrt{\ln(k+1)}} < \dfrac{1}{k\sqrt{\ln k}}$

and $\displaystyle\lim_{k\to\infty}\frac{1}{k\sqrt{\ln k}} = 0$, converges

Consider $\displaystyle\sum_{k=2}^{\infty}\frac{1}{k\sqrt{\ln k}}$. By integral test,

$$\int_2^{\infty}\frac{dx}{x\sqrt{\ln x}} \rightarrow \lim_{k\to\infty}\int_2^{k}\frac{dx}{x\sqrt{\ln x}}$$

Aside: $\displaystyle\int\frac{dx}{x\sqrt{\ln x}}$ $\quad u=\ln x$, $du=\frac{1}{x}dx$ $\quad \displaystyle\int\frac{1}{\sqrt{u}}du = \int u^{-\frac{1}{2}}du$

$2u^{\frac{1}{2}} + C$

$\displaystyle\lim_{k\to\infty}(2\sqrt{\ln k}) = +\infty$, diverges.

Since $\displaystyle\int_2^{\infty}\frac{dx}{x\sqrt{\ln x}}$ diverges, so does $\displaystyle\sum\frac{1}{k\sqrt{\ln k}}$

The alternating series converges conditionally

c) $\sum_{n=1}^{\infty} \frac{\text{Arctan}(4n)}{n}$ By $\underline{LCT}$, $\sum \frac{1}{n}$

(12 ea) $\lim_{n\to\infty} \frac{\text{Arctan}(4n)}{n} \cdot \frac{n}{1} = \frac{\pi}{2} > 0$ and finite

Since $\sum \frac{1}{n}$ diverges, $p=1$, the series $\underline{\text{diverges}}$.

d) $\sum_{n=2}^{\infty} \frac{(-1)^n \sqrt{n-1}}{\sqrt[3]{8n^4-n^3+1}}$ By $\underline{AST}$,

$0 < \frac{\sqrt{n}}{\sqrt[3]{8n^4-n^3+2}} < \frac{\sqrt{n-1}}{\sqrt[3]{8n^4-n^3+1}}$ and $\lim_{n\to\infty} \frac{\sqrt{n-1}}{\sqrt[3]{8n^4-n^3+1}} = 0$, $\underline{\text{Converges}}$

$\sum_{n=2}^{\infty} \frac{\sqrt{n-1}}{\sqrt[3]{8n^4-n^3+1}}$ By CT,

$\frac{\sqrt{n-1}}{\sqrt[3]{8n^4-n^3+1}} > \frac{1}{2n^{5/6}}$, $p = \frac{5}{6}$ diverges

Therefore, the alternating series $\underline{\text{converges conditionally}}$

e) $\sum_{k=1}^{\infty} \frac{2^k}{e^k - k}$

By $\underline{LCT}$, $\left(\frac{2}{e}\right)^k$

$\lim_{k\to\infty} \frac{\cancel{2^k}}{e^k-k} \cdot \frac{e^k}{\cancel{2^k}} = \lim_{k\to\infty} \frac{e^k}{e^k-k} = 1 > 0$ and finite

Since $\sum \left(\frac{2}{e}\right)^k$ converges, geometric $r = \frac{2}{e}$, then the series $\underline{\text{converges}}$